Chapter 9.

Sets and probability.

Before commencing this chapter the reader is advised to spend a few minutes re-reading the brief sections on sets and probability in the *Preliminary Work* at the beginning of this book.

Sets and probability – basic ideas.

The next exercise provides you with some practice in the sets and probability ideas referred to in the preliminary work.

Exercise 9A

Probability.

1. A normal fair six sided die is rolled once.
   With P(X) meaning the probability of event X occurring determine
   (a) P(an even number).
   (b) P(an odd number).
   (c) P(a prime number). Remember: A prime has exactly 2 factors.
   (d) P(a number greater than four).
   (e) P(a number not less than three).

2. Two fair dice are rolled once and the numbers on the uppermost faces are added together. The table on the right shows the 36 equally likely outcomes.
   With P(X) meaning the probability of event X occurring determine
   (a) P(an even total).
   (b) P(an odd total).
   (c) P(a prime total).
   (d) P(a total of 11).
   (e) P(a total that is greater than 8).
   (f) P(a total that is not greater than 8).

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

3. A particular event has just three possible outcomes A, B and C. Only one of these outcomes can occur each time the event happens and P(A) = 0·5 and P(B) = 0·2.
   If the event occurs once determine
   (a) P(C) (b) P(not C) (c) P(not A) (d) P(not B)

4. If a coin is flipped three times there are eight equally likely outcomes:

(H)(H)(H) (H)(H)(T) (H)(T)(H) (T)(H)(H)

(T)(T)(H) (T)(H)(T) (H)(T)(T) (T)(T)(T)

For such an event determine:
(a) P(HHH).
(b) P(Two heads and a tail in that order).
(c) P(Two heads and a tail in any order).
(d) P(The third flip produces a head).
(e) P(The third flip produces the only head).
(f) P(The third flip produces the second head).

5. A fair die is rolled onto a flat wooden table and when the die has come to rest the five numbers that can be seen are added together.
For one such roll determine: (a) P(The total obtained is less than 15).
(b) P(The total obtained is more than 15).
(c) P(The total obtained is divisible by 3).
(d) P(The total obtained is divisible by 5).
(e) P(The total obtained is divisible by both 3 and 5).

6. A child makes a spinner which, on 100 spins produces the results:

Result of spin	1	2	3	4	5	6	7
Frequency	12	10	8	22	20	15	13

Based on these figures what is the probability that on one spin of the spinner the result will be (a) 2, (b) 6, (c) even,
(d) > 4, (e) ≥ 4, (f) < 3.

7. The causes of death given for the sixty seven thousand two hundred and forty one Australian males who died in one particular year were as follows:

Cardiovascular Disease	Cancers	Traffic Accidents	All others
21 957	22 039	1 224	22 021

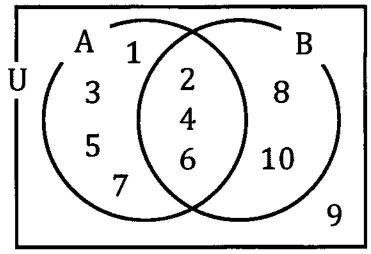
[Source of data: National Heart Foundation of Australia and The Australian Bureau of Statistics.]

An insurance company uses these figures to determine the probable causes of death amongst Australian Males for the following year. What do these figures suggest for the probability of the death of an Australian male being due to
(a) cardiovascular disease, (correct to 3 d.p.)
(b) a cause other than Cancer, (correct to 3 d.p.).

Sets.

8. From the Venn diagram on the right we see that

$$A = \{1, 2, 3, 4, 5, 6, 7\}.$$



(a) State n (A). (b) State n (A ∪ B).
(c) State n (U). (d) State n (A ∩ B).
(e) List the elements of A'.
(f) List the elements of $\overline{B}$.
(g) List the elements of $\overline{A \cup B}$.
(h) List the elements of $\overline{A \cap B}$.

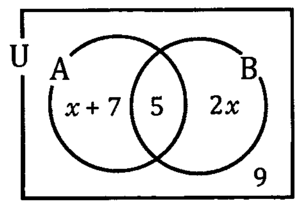
9. The universal set, U, and the two sets A and B contained within it are such that

$$n(A \cap B) = 6 \qquad n(A) = 27$$
$$n(B) = 46 \quad \text{and} \quad n(U) = 70$$

Determine (a) n(A ∪ B) (b) $n(\overline{A \cup B})$

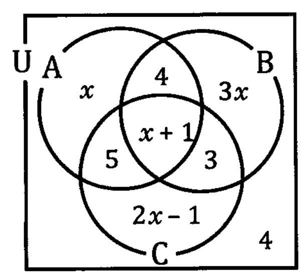
10. Eighty students commenced a particular university course. Of these, 42 had studied Physics in their last year at school and 46 had studied Chemistry in their last year at school. If 25 had studied both of these subjects in their last year at school how many had studied neither?

11. The Venn diagram on the right shows the number of elements in each of sets A and B and in the Universal set, U, in which A and B are contained.
If n(U) = 72 find the value of x.



12. Of the 137 year eleven students in a school the number who had represented the school at volleyball but not athletics was 7 more than the number who had represented the school at both volleyball and athletics.
The number who had represented the school at volleyball but not athletics was twice the number who had represented the school at athletics but not volleyball.
The number who had represented the school at neither volleyball nor athletics was nine times the number who had represented the school at both volleyball and athletics.
How many of the year eleven students had represented the school at both volleyball and athletics?

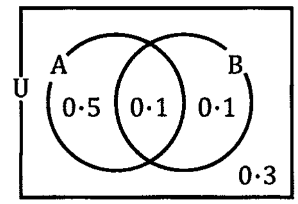
13. The Venn diagram on the right shows the number of elements in each of sets A, B and C and in the Universal set, U, in which A, B and C are contained.
If $n(\overline{B \cup C}) = 11$ find (a) n(U),
(b) n(A ∩ B),
(c) n(A ∩ B ∩ C).



Probability and Sets.

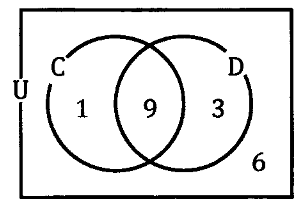
In questions14 to 16 the notation P(X) is used to mean the probability that an element randomly selected from the universal set U is in set X.

14. The Venn diagram shows the probabilities of the events A and B occurring. Find



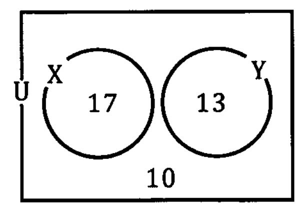
(a) P(A) (b) P(B) (c) $P(A \cap B)$
(d) $P(A \cup B)$ (e) $P(\overline{A \cap B})$ (f) $P(\overline{A \cup B})$

15. The Venn diagram on the right shows how the 19 elements in the universal set U are placed with regards to sets C and D. Determine:



(a) P(C) (b) P(D) (c) $P(C \cap D)$
(d) $P(\overline{C \cap D})$ (e) $P(C \cap \overline{D})$ (f) $P(\overline{C} \cap D)$
(g) $P(C \cup D)$ (h) $P(C \cup \overline{D})$ (i) $P(\overline{C} \cup D)$

16. The Venn diagram on the right shows how the 40 elements in the universal set U are placed with regards to sets X and Y. Determine:



(a) P(X) (b) P(Y) (c) $P(X \cap Y)$
(d) $P(X \cup Y)$ (e) $P(\overline{X \cup Y})$ (f) $P(\overline{X \cap Y})$

17. Two events A and B are such that P(A) = 0·3, P(B) = 0·5 and $P(A \cup B)$ = 0·6, where the notation P(A) means the probability of event A occurring.
Determine $P(A \cap B)$.

18. If a student is randomly selected from the year 12 students in a particular school the probability of that student being male is 0·52, the probability that they study chemistry is 0·44 and the probability of them being female and not doing chemistry is 0·18.
Determine the probability that the selected student is
(a) a female studying chemistry,
(b) a male not studying chemistry.

19. Twenty dancers from a particular dance school attend the national dance championships. All of the twenty are entered in either solo events or team events or both solo and team.
Five of the twenty are not entered in solo events and two are not entered in team events.
(a) If one of the twenty dancers is randomly selected to represent the school in the opening ceremony what is the probability that this dancer is one who is entered in both solo and team events?
(b) One of the dancers is injured when competing in the solo event. What is the probability that this dancer is one who is entered in team events?

Conditional probability.

In some situations we may be *given* some extra piece of information, or some additional *condition*, which allows us to reduce the number of possibilities that we need to consider in the sample space. Indeed the last question in the previous exercise is like this. The information given in part (b) informs us that the dancer we are considering is one who competes in a solo event, and so allows us to consider only the solo dancers.
The following examples and the questions of Exercise 9B give further examples of this idea of *conditional probability.*

Example 1

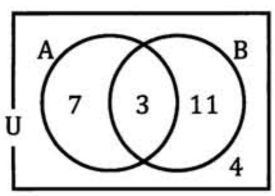
Arika rolls a fair die once and May, who cannot see the result, tries to guess the outcome.
(a) What is the probability that the result is a 5?
Before May states her guess Arika announces "It's an odd number".
(b) Now what is the probability that the result is a 5?

(a) There are six equally likely outcomes: 1, 2, 3, 4, 5, 6.
A result of a 5 is one of these 6. Thus $P(5) = \frac{1}{6}$.

(b) Given the information *It's an odd number* there are now just three equally likely outcomes: 1, 3, 5.
A result of a 5 is one of these 3. Thus $P(5) = \frac{1}{3}$.

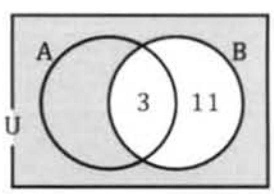
Example 2

The numbers in the various sections of the Venn diagram on the right indicate the number of people in each of the sets A and B. Determine the probability that a person chosen at random from the universal set, U, is
(a) in set B
(b) outside of set B
(c) in set A given they are in set B,
(d) in set A given they are in A ∪ B.

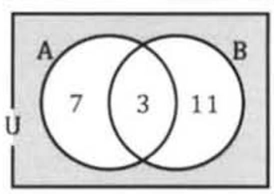


(a) P(person is in set B) = $\frac{14}{25}$ (b) P(outside of set B) = $\frac{11}{25}$

(c) Given that the person is in set B we need only consider the 14 people in B (see diagram).
3 of these 14 are in set A.
Thus P(person is in set A given they are in set B) = $\frac{3}{14}$



(d) Given that the person is in A ∪ B we need only consider the 21 people in A ∪ B (see diagram).
10 of these 21 are in set A.
Thus P(person is in set A given they are in A ∪ B) = $\frac{10}{21}$



Notation

In part (c) of the previous example we were asked for the probability of a person being in set A **given** they are in set B.

We write the probability of some event X occurring **given** some other condition Y as

$$P(X \mid Y)$$

and we refer to this as the probability of X *given* Y.

Thus in part (c) of the last example :

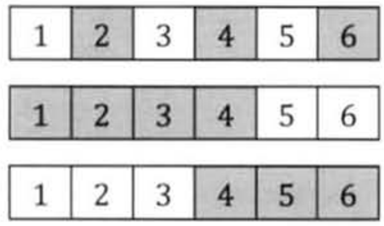
$$P(\text{person is in set A} \mid \text{they are in set B}) = \frac{3}{14}.$$

Check that you agree with the following statements.

For a single roll of a fair die: $P(5 \mid \text{odd number}) = \frac{1}{3}$,

$P(5 \mid \text{greater than } 4) = \frac{1}{2}$,

$P(5 \mid \text{less than } 4) = 0$.



Example 3

A fair octahedral die is rolled once. Determine

(a) P(7), (b) P(7 | odd number), (c) P(3 | prime number),

(d) P(not a 2 | a number < 4), (e) P(odd number | 7).

(a) On an octahedral die there are eight equally likely outcomes: 1, 2, 3, 4, 5, 6, 7, 8.

A result of a 7 is one of these 8 equally likely outcomes. Thus $P(7) = \frac{1}{8}$.

(b) Given that the result is an odd number there are now just four equally likely outcomes: 1, 3, 5, 7.

A result of a 7 is one of these 4 equally likely outcomes.

Thus: $P(7 \mid \text{odd number}) = \frac{1}{4}$.

(c) Given that the result is a prime number there are just four equally likely outcomes: 2, 3, 5, 7.

A result of a 3 is one of these 4 equally likely outcomes.

Thus: $P(3 \mid \text{prime number}) = \frac{1}{4}$.

(d) Given that the result is a number less than 4 there are just three equally likely outcomes: 1, 2, 3.

Two of these three equally likely outcomes are "not a 2".

Thus: $P(\text{not a } 2 \mid \text{a number} < 4) = \frac{2}{3}$.

(e) Given that the result is a 7 there is just one outcome, a 7, and this is an odd number.

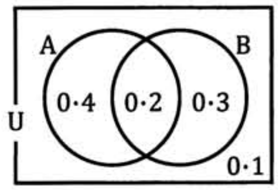
Thus: $P(\text{odd number} \mid 7) = \frac{1}{1}$

$= 1.$

Example 4

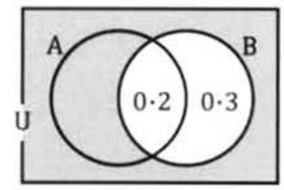
The Venn diagram on the right shows the probabilities of events A and B occurring.

Determine (a) $P(A \cap B)$ (b) $P(A \mid B)$ (c) $P(B \mid \overline{A})$



(a) $P(A \cap B) = 0{\cdot}2$

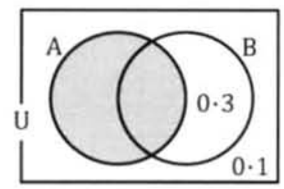
(b) Given that event B occurs we need only consider that part of the Venn diagram (see unshaded part on right).



The total under consideration is now 0·5.
0·2 lies in circle A.

$$\text{Thus } P(A \mid B) = \frac{0{\cdot}2}{0{\cdot}5} = 0{\cdot}4$$

(c) Given that $\overline{A}$ occurs we need only consider that part of the Venn diagram (see unshaded part on right).



The total under consideration is now 0·4.
0·3 lies in circle B.

$$\text{Thus } P(B \mid \overline{A}) = \frac{0{\cdot}3}{0{\cdot}4} = 0{\cdot}75$$

Exercise 9B

1. Jack rolls a normal fair die once and Holly, who cannot see the result, tries to guess the outcome.
   (a) What is the probability that the result is a 4?
   Before Holly states her guess Jack announces "It's not a six".
   (b) Now what is the probability that the result is a 4?

2. Ⓗ Ⓗ   Ⓗ Ⓣ   Ⓣ Ⓗ   Ⓣ Ⓣ

   Leroy tosses two fair coins and Boon attempts to guess the result.
   (a) What is the probability that the result is two heads?
   Before Boon announces his guess Leroy states "It's not two tails".
   (b) Now what is the probability that the result is two heads?

3. Leslie tosses two fair normal six sided dice, one red and one blue, and Ranji attempts to guess the total obtained when the number of dots on each uppermost face are added.
   (a) What is the probability that the total is eleven?

   In fact Ranji manages to see that the red die lands with a six uppermost.
   (b) Now what is the probability that the total is eleven?

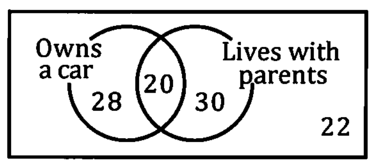
BLUE DIE \ RED DIE	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

4. Mansur randomly selects a card from a normal well shuffled pack of 52 playing cards and Marlena attempts to guess what card has been selected.

Hearts (red)	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
Diamonds (red)	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Spades (black)	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Clubs (black)	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣

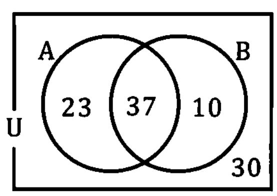
   (a) What is the probability that the card is the two of hearts?
   (b) If in fact Marlena manages to see that the card is red and is not a Jack, King or Queen what is the probability now that it is the two of hearts?

5. The Venn diagram on the right displays the results of a survey of 100 university students regarding whether they own a car and if they are living with their parents.



   (a) What is the probability that a student chosen at random from this group does own a car but does not live with their parents?
   (b) Given that a student chosen from this group does not own a car what is the probability that they live with their parents?

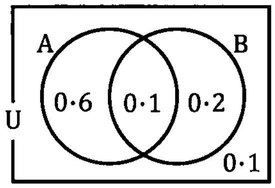
6. The numbers in the various sections of the Venn diagram on the right indicate the number of people in each of the sets A and B. If one person is chosen at random from the universal set, U, determine the probability they are in



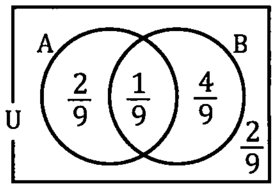
   (a) set A, (b) set B, (c) $A \cup B$,
   (d) $A \cap B$, (e) $\overline{A}$, (f) $\overline{B}$,
   (g) set A given they are in set B, (h) set A given they are not in set B.

7. A fair die is rolled once.
   Event A is that of the result of the roll being a 6.
   Event B is that of the result of the roll being bigger than 4.
   Event C is that of the result of the roll being an even number.
   Determine (a) $P(A \mid B)$ (b) $P(A \mid C)$ (c) $P(B \mid C)$
   (d) $P(B \mid A)$ (e) $P(C \mid A)$ (f) $P(C \mid B)$

8. The Venn diagram on the right shows the probabilities of the events A and B occurring. Find
   (a) P(A) (b) P(B) (c) P(A ∪ B)
   (d) P($\overline{A}$) (e) P($\overline{B}$) (f) P(A|B)
   (g) P(B | A) (h) P(A | A ∪ B) (i) P(B | A ∩ B)



9. The Venn diagram on the right shows the probabilities of the events A and B occurring. Find
   (a) P(A) (b) P(B) (c) P(B ∪ A)
   (d) P(A ∩ B) (e) P($\overline{A}$) (f) P($\overline{B}$)
   (g) P(A | B) (h) P(B | A) (i) P(B | A ∪ B)



10. A team of two people is to be randomly selected from the following five people:
    Alex, Basil, Chris, Deny, Ernie.

    There are ten different teams of two that can be made:

Alex	Alex	Alex	Alex	Basil	Basil	Basil	Chris	Chris	Deny
Basil	Chris	Deny	Ernie	Chris	Deny	Ernie	Deny	Ernie	Ernie

   (a) Find the probability that Alex is selected.
   (b) Find the probability that the team of two consists of Deny and Alex.
   (c) Given that Deny is selected what is the probability that Alex is also selected?
   (d) What is the probability that just one of Deny and Alex is selected?
   (e) Given that just one of Deny and Alex is selected what is the probability that Ernie fills the other space on the team?

11. A fair normal six sided die is rolled once. Determine
   (a) P(3), (b) P(3 | odd number),
   (c) P(3 | even number), (d) P(even number | 3),
   (e) P(2 | a number < 4), (f) P(not a 2 | a number < 4).
   (g) P(a number < 2 | a number < 4), (h) P(a number ≤ 2 | a number < 4).

12. A fair ten sided die, with faces marked 1 to 10, is rolled once. Determine
   (a) P(8), (b) P(8 | even number),
   (c) P(8 | prime number), (d) P(7 | prime number),
   (e) P(8 | a number > 4), (f) P(not an 8 | a number > 4).
   (g) P(a number > 8 | a number ≥ 6).

13. From a set of cards numbered 1 to 20 one card is selected at random. What is the probability that the card is
   (a) a seven given that it is less than ten,
   (b) a seven given that it is more than ten,
   (c) a six given that it is a multiple of three,
   (d) a six given that it is a factor of twelve,
   (e) an even number given that it is neither four nor ten?

14. A normal six sided die is rolled twice and the scores obtained are added together.
    Event A is that of the first roll giving a 5.
    Event B is that of the total being 10.
    Determine (a) P(B | A), (b) P(A | B).
    Event C is that of the first roll giving a 2.
    Event D is that of the total being 6.
    Determine (c) P(D | C), (d) P(C | D).
    Event E is that of the first roll giving an odd number.
    Event F is that of the total being 5.
    Determine (e) P(F | E), (f) P(E | F).

FIRST ROLL \ SECOND ROLL	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

15. A fair icosahedral die (20 faces numbered 1 to 20) is rolled once. Determine
    (a) P(3)
    (b) P(even)
    (c) P(prime)
    (d) P(multiple of 3)
    (e) P(factor of 12)
    (f) P(5 | an odd number)
    (g) P(3 | < 6)
    (h) P(15 | > 9)
    (i) P(9 | a multiple of 3)
    (j) P(a multiple of 3 | a factor of 12)
    (k) P(multiple of 3 | > 15)
    (l) P(multiple of 3 | ≥ 15)

16. The twenty four 3-digit numbers that can be formed using the digits 1, 2, 3, and 5, with each digit being used only once in each number are as follows:

123	132	213	231	312	321
235	253	325	352	523	532
135	153	315	351	513	531
125	152	215	251	512	521

One of these twenty four numbers is to be selected at random.
Event A is that of the selected number being bigger than 300.
Event B is that of the selected number being bigger than 400.
Determine (a) P(A) (b) P(B) (c) P(A | B)
(d) P(B | A) (e) $P(A \mid \overline{B})$ (f) $P(\overline{A} \mid B)$

17. (Hint: A little thought could save you time with this question.)
    A fair coin is tossed four times. What is the probability that the result of the next throw, i.e. the fifth throw, will be a head given that
    (a) the fourth throw resulted in a head,
    (b) the third and the fourth throws resulted in heads.

Research and write a brief report on:

The Gambler's fallacy.

Tree diagrams.

As included in the brief mention of probability in the preliminary work at the start of this chapter, another useful form of presentation when finding probabilities is a tree diagram. The next example demonstrates the use of this method of displaying the equally likely outcomes of an event. The last two parts of the example also involve conditional probability.

Example 5

Envelope I contains the three letters: A, B and C.

Envelope II contains the three letters: B, C and D.

A two letter "word" is formed, the first letter being randomly chosen from envelope I and the second letter randomly chosen from envelope II.

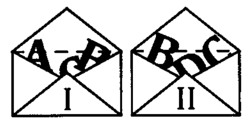
The tree diagram on the right shows the 9 equally likely outcomes. Determine the probability that

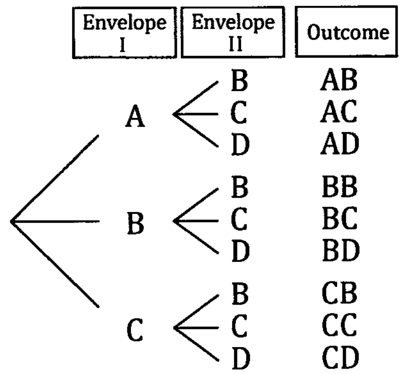
(a) exactly one letter B is chosen,

(b) the letters chosen are the same,

(c) the first letter chosen is an A given that the two letters are not the same,

(d) the second letter chosen is a B given the two letters are the same.





(a) Four of the nine equally likely outcomes contain exactly one B.

P(Exactly one B) $= \frac{4}{9}$

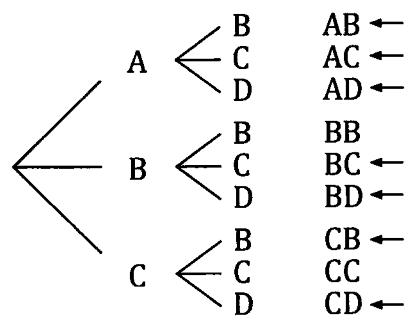
(b) In two of the nine equally likely outcomes the letters are the same.

P(Same letters) $= \frac{2}{9}$

(c) Given that the two letters are not the same we need only consider those outcomes shown arrowed in the tree diagram on the right.

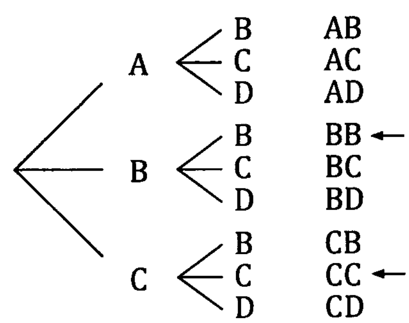
Three of these seven outcomes start with A

P(Start with A | not same letters) $= \frac{3}{7}$



(d) Given that the two letters are the same we only consider those outcomes shown arrowed in the tree diagram on the right.

P(Second letter B | same letters) $= \frac{1}{2}$



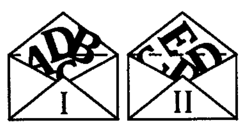
Exercise 9C.

1. Two marbles are drawn at random from a bag containing four marbles - 2 red, 1 blue and 1 green. After the first marble is drawn and its colour noted it is returned to the bag before the second marble is drawn. Construct a suitable tree diagram and hence determine the following probabilities:
   (a) P(two reds | first one red) (b) P(two reds | both same colour)
   (c) P(two different | 2nd one green) (d) P(no reds | 2nd one green)

2. Two marbles are drawn at random from a bag containing four marbles - 2 red, 1 blue and 1 green. After the first marble is drawn and its colour noted it is **not** returned to the bag.
   Construct a suitable tree diagram and hence determine the following probabilities:
   (a) P(two reds | first red) (b) P(one green | first not blue)
   (c) P(first blue | second red) (d) P(second red | first blue)

3. Laurie, Rob and Steven play two rounds of a game of cards. The game has no draws, ties or stalemates so each round is won by one of the three people and you may assume that each player has the same chance of winning each round.
   Construct a suitable tree diagram and hence determine the probability of each of the following:
   (a) Laurie wins both rounds,
   (b) Rob wins at least one of the rounds,
   (c) Laurie wins neither given that Steven wins the second round,
   (d) Steven wins the second round given that Laurie wins neither.

4. Envelope I contains 4 letters: 1 A, 1 B, 1 C and 1 D.
   Envelope II contains 4 letters: 1 C, 2 Ds and an E.
   A two letter "word" is formed, the first letter being randomly chosen from envelope I and the second letter randomly chosen from envelope II.



   Construct a suitable tree diagram and hence determine the probability that:
   (a) one of the two letters is an A, (b) the letters chosen are the same,
   (c) the first letter chosen is a D given that the two letters are the same,
   (d) the second letter chosen is a D given the two letters are not the same.

5. The five letters of the word EXACT are written on five cards, with one letter on each card. The five cards are then shuffled and two of the cards are dealt face up in a line to form a "word".
   Construct a suitable tree diagram and hence determine the probability that the "word" so formed
   (a) is the word AT, (b) starts with an E,
   (c) ends with a T, (d) starts with an E and ends with a T,
   (e) starts with an E given that it ends with a T,
   (f) starts with a T given that it ends with an E,
   (g) contains an X given that it contains an A.

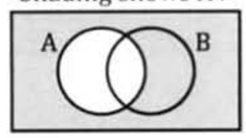
Use of the word "not" in probability questions.

As was mentioned in the preliminary work, prior to chapter one:
If the probability of an event occurring is "a" then the probability of it **not** occurring is

$$1 - a.$$

Thus the phrase "not A" in probability is similar to
A', the complement of A, in sets.

Shading shows A'.

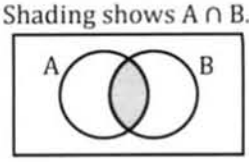


Use of the word "and" in probability questions.

If we require the probability that events A **and** B occur we must look at those situations in which <u>both</u> A and B occur.

Thus the phrase "A and B" in probability is similar to
A ∩ B, in sets.

Shading shows A ∩ B.



Use of the word "or" in probability questions.

Consider the following situations:

☛ Young Jim notices two bars of chocolate on the kitchen work surface.
Jim: *"Mum, can I have a bar of chocolate?"*
Mum: *"Yes dear, you can have the Mars Bar* ***or*** *the Kit Kat."*
Do you think Jim's mum means him to have <u>both</u> bars?

☛ Ms Swift, a deputy principal in XYZ high school, makes the following announcement over the school public address to all year 11 classes:
Ms Swift *"I would like any students doing year 11 Physics* ***or*** *year 11 Art to attend an important meeting in room 3 at the beginning of lunchtime."*
Toni Collinge is a student at XYZ and does <u>both</u> year 11 Physics and year 11 Art. Do you think Ms Swift expects Toni to attend?

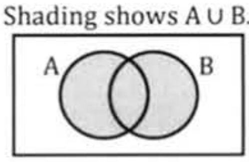
In the first situation Jim's mum would probably not be at all pleased if Jim had both bars. Her use of the word "or" probably meant "one or the other but not both".

In the second situation Ms Swift needed to meet with any student doing Physics or Art. Her use of the word "or" probably meant "one or the other or both".
In these two situations it is only our common sense understanding of the context and the likely intention behind the use of the word "or" that allows us to interpret "or" to mean "one or the other but not both" in one situation
and "one or the other or both" in the other.

To avoid this possible confusion, when the word "or" is used in mathematics we interpret "A or B" to mean "one or the other or both". Hence in probability questions the word "or" should be taken as including the possibility of both i.e. it can be interpreted as meaning "at least one of".

Thus the phrase "A or B" in probability is similar to
A ∪ B in sets.
(Note how the use of ∪ and ∩ in sets avoids the problem.)

Shading shows A ∪ B.



Example 6

The table on the right shows the 36 equally likely number pair outcomes for rolling one red die and one blue die.
Determine the probability that one roll of the two dice will result in:
(a) a 3 on the blue and a 5 on the red,
(b) a 3 on the blue or a 5 on the red,
(c) neither die showing a 5.

	BLUE DIE 1	2	3	4	5	6
RED DIE 1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(a) One of the 36 number pairs, (5, 3), is a 3 on the blue and a 5 on the red.

P(a 3 on the blue and a 5 on the red) $= \frac{1}{36}$

(b) 11 of the 36 number pairs involve a 3 on the blue or a 5 on the red.

P(a 3 on the blue or a 5 on the red) $= \frac{11}{36}$

(c) In 25 of the 36 number pairs neither die shows a 5.

P (neither die showing a 5) $= \frac{25}{36}$

Exercise 9D

1. The table on the right shows the thirty-six equally likely number pair outcomes for rolling one red die and one blue die.
Determine the probability that one roll of the two dice will result in:
(a) not getting a 3 on the blue die,
(b) a 5 on the red and a 1 on the blue,
(c) a 1 on the red or a 5 on the blue,
(d) a 4 on the red and a number bigger than 4 on the blue,
(e) a 4 on the red or a number bigger than 4 on the blue.

	BLUE DIE 1	2	3	4	5	6
RED DIE 1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

2. The table below shows the 12 equally likely outcomes for rolling a fair die and tossing a fair coin.

	DIE 1	2	3	4	5	6
COIN Head	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tail	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

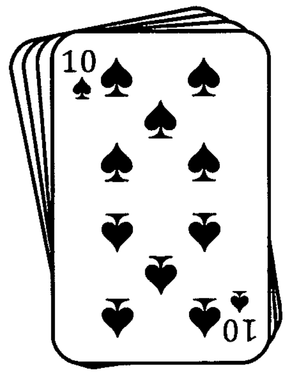
Determine the probability that when the die is rolled and the coin is tossed the result will be
(a) a head on the coin,
(b) a two on the die,
(c) a tail on the coin and a 6 on the die,
(d) a tail on the coin or a 6 on the die,
(e) a head on the coin and an odd number on the die,
(f) either a 6 on the die or a head on the coin but not both of these things.

3. A normal pack of 52 cards is shuffled and a card is selected at random.
The 52 equally likely outcomes are shown below:

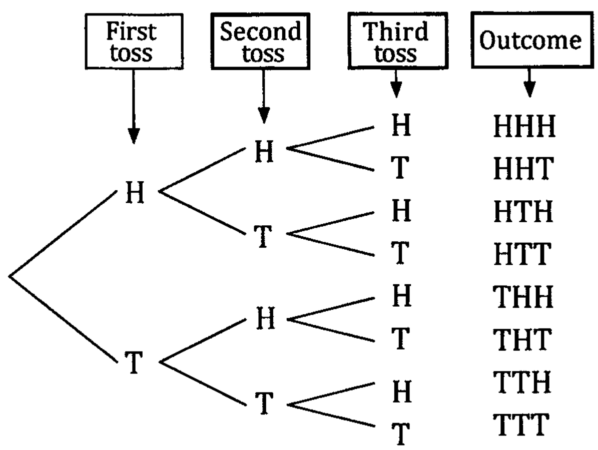
Hearts (red)	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
Diamonds (red)	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Spades (black)	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Clubs (black)	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣

If one card is selected at random from the shuffled pack determine the probability that the card is

(a) the queen of diamonds,
(b) a queen,
(c) a red card,
(d) a three,
(e) a heart,
(f) a jack,
(g) not a jack,
(h) a jack, queen or king,
(i) a red nine,
(j) a red or a nine,
(k) the queen of hearts,
(l) a queen or a heart.



4. The tree diagram below shows the eight equally likely outcomes that could result when a fair coin is tossed three times.



Determine the probability that when a coin is tossed three times the outcome is:
(a) a tail last,
(b) a head first and a tail last,
(c) a head first or a tail last,
(d) the same result on the third toss as was obtained on the second toss.

Tree diagram showing probabilities.

Consider the following question:

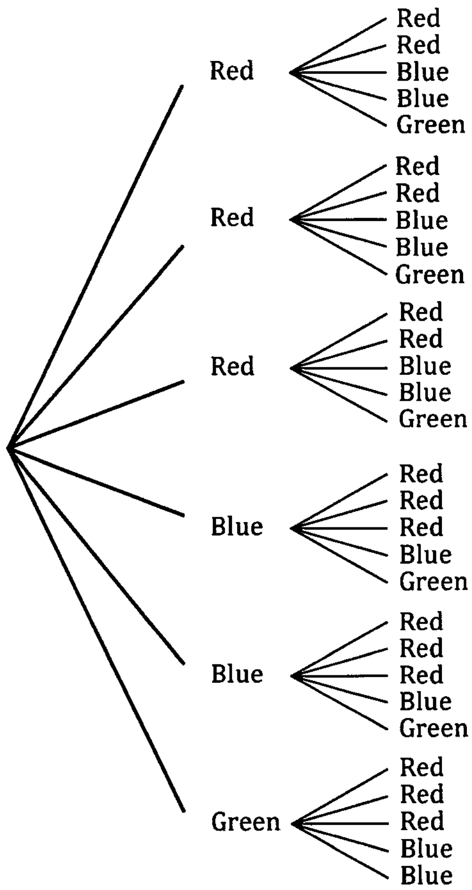
A bag contains six marbles: 3 red, 2 blue and 1 green. Two marbles are randomly selected from the bag, one after the other, the first marble not being replaced before the second is selected. Determine the following probabilities

(a) P(red and blue in that order)

(b) P(two marbles of the same colour)

(c) P(blue first | blue second)

The tree diagram, though rather large, can be constructed and the required probabilities determined:



(a) There are thirty equally likely outcomes.

Six of these thirty are "red then blue".

$$P(\text{red and blue in that order}) = \frac{6}{30} = \frac{1}{5}$$

(b) There are thirty equally likely outcomes.

Two marbles of the same colour occur on eight of these.

$$P(\text{two of the same colour}) = \frac{8}{30} = \frac{4}{15}$$

(c) Given the second marble was blue we only need to consider the ten outcomes for which that is the case.

The first marble being blue occurs on two of these ten.

$$P(\text{blue first} \mid \text{blue second}) = \frac{2}{10} = \frac{1}{5}$$

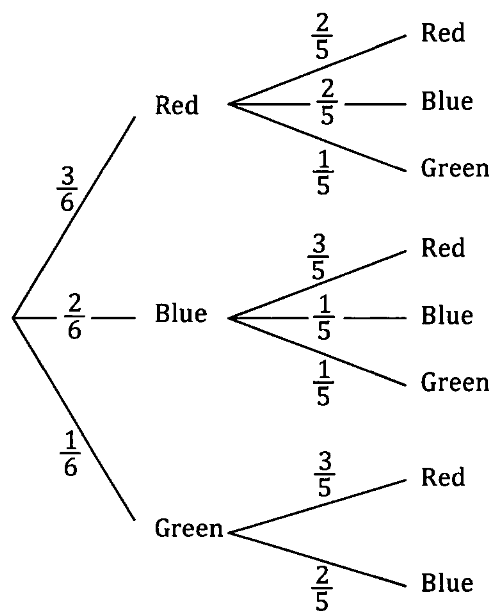
Our tree diagram shows six initial branches, because there are six equally likely outcomes for the first marble out of the bag. A smaller, more manageable tree diagram can be produced by having just three branches to start with, one for each outcome, red, blue, green. We then show the probability of each outcome on the relevant branch of the diagram.

A diagram of this form, for the above situation, is shown on the next page.

Check carefully that you agree with the probability that is on each branch of the following diagram. Remember the bag initially contains 6 marbles in all, 3 red, 2 blue and 1 green, and the first marble drawn is not being replaced before the second is drawn.

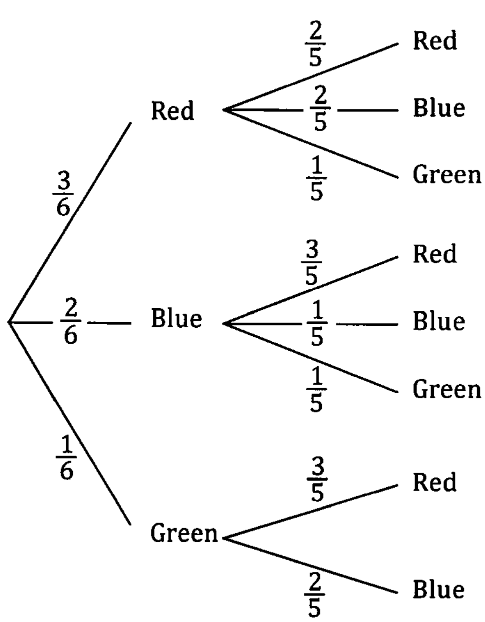
Note the following points:

- In a tree diagram it is important that as we move right, the branches leaving each junction are **mutually exclusive** (if one occurs the others cannot) and cover **all** eventualities that can happen from that point (are **exhaustive**). Thus the probabilities on the branches going right from a single point will sum to 1.
  For example, in the tree diagram shown,
  $\frac{3}{6}+\frac{2}{6}+\frac{1}{6}=1,\qquad \frac{2}{5}+\frac{2}{5}+\frac{1}{5}=1,$
  $\frac{3}{5}+\frac{1}{5}+\frac{1}{5}=1,\qquad \frac{3}{5}+\frac{2}{5}=1.$



- The probabilities on each branch are assigned with due regard to what has happened up to that point.
  For example, if a red is drawn first the probability of the second marble being red is $\frac{2}{5}$ but if the first is blue then the probability of the second being red is $\frac{3}{5}$.

The probability of each final outcome is obtained by following the branches to that outcome, and multiplying probabilities along that route:



$P(RR)=\frac{3}{6}\times\frac{2}{5}=\frac{6}{30}$

$P(RB)=\frac{3}{6}\times\frac{2}{5}=\frac{6}{30}$

$P(RG)=\frac{3}{6}\times\frac{1}{5}=\frac{3}{30}$

$P(BR)=\frac{2}{6}\times\frac{3}{5}=\frac{6}{30}$

$P(BB)=\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$

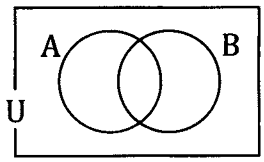
$P(BG)=\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$

$P(GR)=\frac{1}{6}\times\frac{3}{5}=\frac{3}{30}$

$P(GB)=\frac{1}{6}\times\frac{2}{5}=\frac{2}{30}$

This multiplication of probabilities should seem a reasonable thing to do because each new branch involves a fraction of the probability that applied up to that point. It can be further justified using sets, as shown below.

Consider the Venn diagram on the right.
To determine P(B | A) we first consider P(A), because we know event A occurs, and then see how much of P(A) involves B occurring.



i.e. $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Rearranging: $$P(A \cap B) = P(A) \times P(B \mid A)$$

This last statement is the *multiplication rule* for probabilities. It tells us that to determine P(both A and B occurring) we multiply the probability of A by the probability of B given A. This is exactly what we do when we multiply probabilities as we pass along the branches of a tree diagram towards a final outcome.

In the tree diagram we can combine final outcomes by adding probabilities.

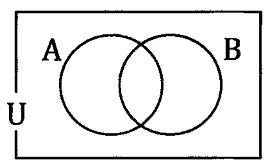
For example: P(both marbles the same colour) = P(RR) + P(BB)

$$= \frac{6}{30} + \frac{2}{30} = \frac{4}{15}$$

Whilst this addition of probabilities should also seem a reasonable thing to do, it can be justified using sets:

Now $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\therefore \quad \frac{n(A \cup B)}{n(U)} = \frac{n(A)}{n(U)} + \frac{n(B)}{n(U)} - \frac{n(A \cap B)}{n(U)}$$



i.e. $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

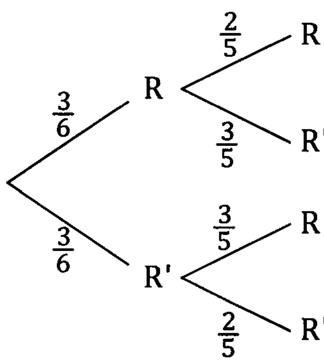
This last statement is the *addition rule* for probabilities. If A and B are *mutually exclusive events* (cannot occur together) it follows that $P(A \cap B) = 0$ and the rule reduces to:

$$P(A \cup B) = P(A) + P(B)$$

This tells us that to determine P(A or B will occur), where A and B are mutually exclusive events, we add the separate probabilities. In a tree diagram the final outcomes are mutually exclusive so we can combine them in this way.

Note
- We will see the multiplication rule and the addition rule again later in this chapter. For the moment our tree diagram approach is allowing us to apply the rules intuitively.
- As we would expect, the final probabilities in the tree diagram sum to 1:
  $$\frac{6}{30} + \frac{6}{30} + \frac{3}{30} + \frac{6}{30} + \frac{2}{30} + \frac{2}{30} + \frac{3}{30} + \frac{2}{30} = \frac{30}{30} = 1$$
- The probabilities in the tree diagram are sometimes initially best left "uncancelled" to make combining them easier.

- If all we are concerned with in the marbles in the bag situation is the probability of getting (or not getting) reds, the tree diagram could be further simplified as shown on the right.
  In this diagram R' is used to represent the event of the selected marble not being red.



$P(RR) = \frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

$P(RR') = \frac{3}{6} \times \frac{3}{5} = \frac{9}{30}$

$P(R'R) = \frac{3}{6} \times \frac{3}{5} = \frac{9}{30}$

$P(R'R') = \frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

The following examples show the use of this idea of a tree diagram with probabilities shown on the branches and the intuitive use of the rules for the multiplication and addition of probabilities.

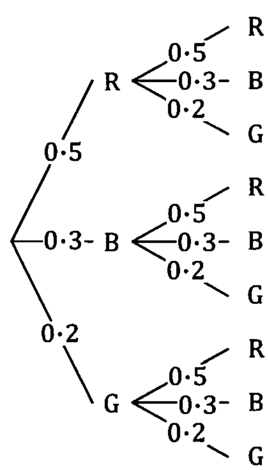
Example 7

A bag contains ten marbles: 5 red, 3 blue and 2 green. Two marbles are randomly selected from the bag, one after the other, the first marble being replaced before the second is selected. Determine the following probabilities

(a) P(red and blue in that order) (b) P(red and blue in any order)

(c) P(two marbles of the same colour) (d) P(two reds|both same colour)

First draw a tree diagram:



P(RR) = 0·25
P(RB) = 0·15
P(RG) = 0·1
P(BR) = 0·15
P(BB) = 0·09
P(BG) = 0·06
P(GR) = 0·1
P(GB) = 0·06
P(GG) = 0·04

(a) P(R then B) = 0·15

(b) P(R and B in any order) = P(RB) + P(BR)
= 0·15 + 0·15
= 0·3

(c) P(same colour) = P(RR) + P(BB) + P(GG)
= 0·25 + 0·09 + 0·04
= 0·38

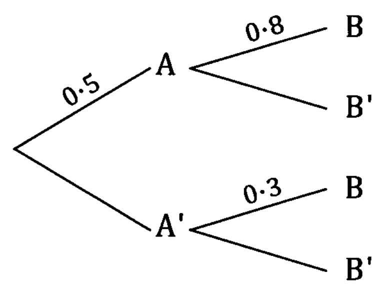
(d) Given that both marbles are the same colour we only consider those events. They have probability 0·38 and amongst this 0·38, 0·25 is when two reds occur.

$\therefore$ P(RR|same colour) $= \frac{0{\cdot}25}{0{\cdot}38}$

$\approx 0{\cdot}66$

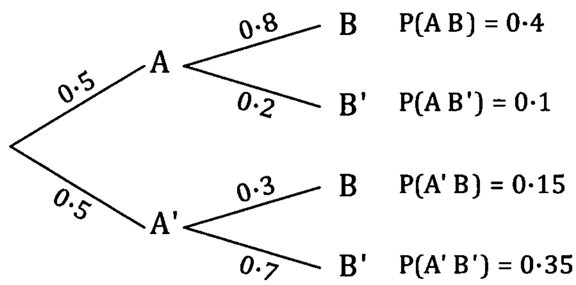
Example 8

The probability event A occurs is 0·5. If event A occurs, the probability of event B occurring is 0·8.
If event A does not occur then the probability of event B occurring is 0·3.
These probabilities are shown in the tree diagram on the right with A' and B' representing the non occurrence of event A and the non occurrence of event B respectively.



Determine (a) P(A')
(b) P(A and B occurring)
(c) P(B)
(d) P(B' | A)
(e) P(A | B)

First complete the tree diagram:



(a) $P(A') = 0{\cdot}5$

(b) $P(A \text{ and } B) = 0{\cdot}4$

(c) $P(B) = 0{\cdot}4 + 0{\cdot}15$
$= 0{\cdot}55$

(d) P(B' | A) is the 0·2 in the tree diagram.
$\therefore \quad P(B' \mid A) = 0{\cdot}2.$
Alternatively, had we not realised this, the same answer can be obtained from the final column of our tree diagram:
Given that A occurs we only consider those events in the final column in which A occurs. These have probability of 0·5 (= 0·4 + 0·1) and amongst this 0·5, 0·1 is for when B' occurs

$\therefore \quad P(B' \mid A) = \dfrac{0{\cdot}1}{0{\cdot}5} = 0{\cdot}2$ as before.

(e) Given that B occurs we only consider those events in the final column in which B occurs. These have probability of 0·55 (= 0·4 + 0·15) and amongst this 0·55, 0·4 is for when A occurs

$\therefore \quad P(A \mid B) = \dfrac{0{\cdot}4}{0{\cdot}55} = \dfrac{8}{11}.$

Example 9

Let us suppose that in a certain breed of chicken, 5% of the chickens are carriers of a particular disease.

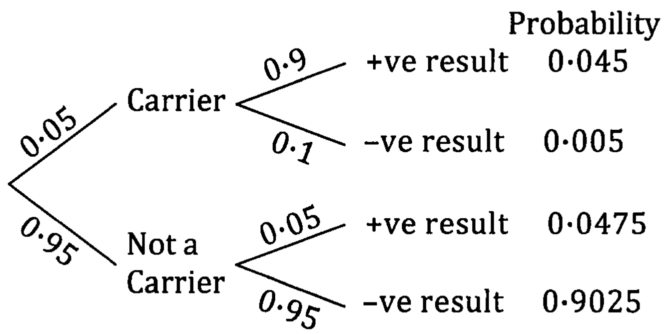
A test that can accurately detect whether or not a chicken is a carrier of the disease is available but it is expensive to administer.

A cheaper test is developed and trials indicate that this returns a positive result in 90% of the chickens who are carriers of the disease, but unfortunately returns a negative result in the other 10%. When the test is administered to chickens known not to be carriers the success rate is 95%. i.e. In 95% of the non carriers a negative result is returned but for the other 5% a positive result is returned.

(a) A chicken from this breed is selected at random and given this cheaper test. What is the probability the test will return a correct result?

(b) A chicken from this breed of chickens is selected at random, given this cheaper test, and returns a positive result. What is the probability the chicken is not a carrier of the disease, despite this positive result?

The tree diagram is as shown:



(a) P(correct result) $= 0{\cdot}045 + 0{\cdot}9025$

$= 0{\cdot}9475$

(b) We require:

P(not a carrier | +ve result)

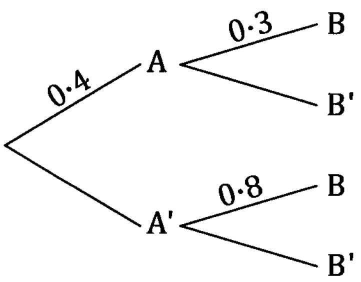
Those outcomes that involve a positive result have probability of 0·0925 (= 0·045 + 0·0475) and these outcomes contain one with probability 0·0475 that involves non carriers.

$$\therefore \quad \text{P(not a carrier | +ve result)} = \frac{0{\cdot}0475}{0{\cdot}0925} \approx 0{\cdot}514$$

You may find this last result interesting. It shows that for the given test, any chicken returning a positive test is still more likely not to be a carrier of the disease than to be a carrier.

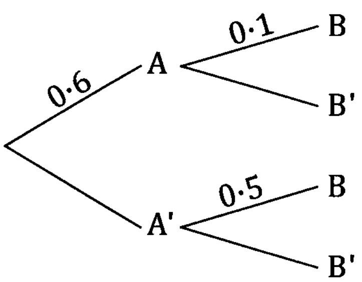
Exercise 9E

1. The probability event A occurs is 0·4.
   If event A occurs, the probability of event B occurring is 0·3.
   If event A does not occur then the probability of event B occurring is 0·8.
   These probabilities are shown in the tree diagram on the right with A' and B' representing the non occurrence of event A and the non occurrence of event B respectively.



   Determine (a) P(A') (b) P(A and B occurring)
   (c) P(B) (d) P(B|A)
   (e) P(A|B) (f) P(A|B')

2. The probability event A occurs is 0·6.
   If event A occurs, the probability of event B occurring is 0·1.
   If event A does not occur then the probability of event B occurring is 0·5.
   These probabilities are shown in the tree diagram on the right with A' and B' representing the non occurrence of event A and the non occurrence of event B respectively.



   Determine (a) P(A')
   (b) P(neither A nor B occurring)
   (c) P(B) (d) P(at least one of A or B occurring)
   (e) P(B|A) (f) P(A|B)

3. A bag contains ten marbles: 7 blue and 3 green. Two marbles are randomly selected from the bag, one after the other, the first marble <u>not</u> being replaced before the second is selected.
   Determine the probability that the two marbles are
   (a) blue and green in that order,
   (b) blue and green in any order,
   (c) of the same colour,
   (d) two blues given they are of the same colour.

4. Bag A contains five marbles: two red and three blue.
   Bag B contains five marbles: one red and four blue.
   Two coins are tossed. If both coins show heads, a marble is randomly selected from those in bag A, otherwise a marble is randomly selected from bag B. Determine the probability that
   (a) the selected marble comes from bag B,
   (b) a blue marble is selected, given the marble selected comes from bag B,
   (c) the marble selected comes from bag A, given that it was a blue marble.

5. Forty per cent of the students at a particular college are male. Eighty per cent of the males and 40% of the females are studying course A.

   If one student is randomly selected from the students at this college determine the probability that the student is

   (a) female,

   (b) studying course A,

   (c) male given they are studying course A,

   (d) female given they are not studying course A.

6. Bag A contains 3 marbles: one red and two blue.
   Bag B contains 4 marbles: one red and three blue.
   Bag C contains 5 marbles: one red and four blue.

   A normal fair die is rolled and if the uppermost face shows 1, 2 or 3 a marble is randomly chosen from bag A, if the die shows 4 or 5 on its uppermost face a marble is randomly chosen from bag B and a 6 on the uppermost face of the die sees a marble randomly drawn from bag C.

   Determine the probability that the selected marble is

   (a) from bag A,

   (b) from bag C,

   (c) red,

   (d) blue,

   (e) red and from bag A,

   (f) blue or from bag B,

   (g) from bag A given that the marble was red,

   (h) from bag B given that the marble was blue.

7. On a particular course, all students who satisfactorily complete all coursework, assignments and tests then sit exam I.

   Those with a mark $\geq 60\%$ in exam I go on to take exam IIA.
   Those with a mark $< 60\%$ in exam I then take exam IIB.
   Two thirds of those taking exam I achieve a mark $\geq 60\%$.
   In exam IIA, one third of those taking it achieve a mark $\geq 70\%$.
   In exam IIB, one quarter of those taking it achieved a mark $< 45\%$.
   Grades are awarded as follows:
   Achieve a mark $\geq 70\%$ in exam IIA: Grade A.
   Achieve a mark $< 70\%$ in exam IIA: Grade B+.
   Achieve a mark $\geq 45\%$ in exam IIB: Grade B−.
   Achieve a mark $< 45\%$ in exam IIB: Grade C.

   One student is selected at random from all of the students completing this course, to represent the students at a function.

   (a) What is the probability that the student selected got a B grade?

   (b) Given that the student selected got a B what is the probability that this was in fact a B+?

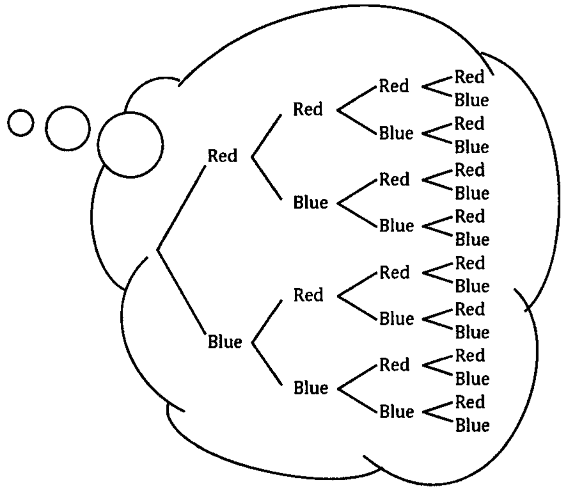
8. A college runs diploma courses each lasting 2 years. The students on these courses are classified either as *first year students* or as *second year students* (but not both). The ratio of first year students to second year students is 5 : 4. Fifty five percent of the first year students, and thirty five percent of the second year students, live at home with their parents. If a student is chosen at random from those at this college on these diploma courses determine the probability that the student is
   (a) a second year student,
   (b) a second year student or a student living at home with their parents,
   (c) a first year student, given they are living at home with their parents.

9. Five per cent of the people in a particular high risk category are thought to have a particular disease. In an attempt to detect the disease in its early stages a test is developed to identify those who have it.
   For those people in the high risk category who do have the disease, the test shows a 98% success rate, i.e. for those who do have the disease the test returns a positive result 98% of the time, and in just 2% the test wrongly returns a negative result (wrongly suggesting that these 2% do not have the disease when in fact they have it).
   For those people in the high risk category who do not have the disease the test shows a 96% success rate, i.e. for those who do not have the disease the test returns a negative result 96% of the time, and in just 4% the test wrongly returns a positive result (wrongly suggesting that these 4% do have the disease when in fact they do not have it).
   Determine the probability that a person selected at random from those in the high risk category who have the test:
   (a) does not have the disease and returns a negative result in the test.
   (b) does not have the disease but returns a positive result in the test.
   (c) returns an incorrect result in the test.
   A person in this high risk category has the test and receives the news that the test gave a positive result.
   (d) Determine the probability that this person really does have the disease. (Give your answer correct to three decimal places.)

10. A bag contains ten discs, indistinguishable except for their colour. Four of the discs are white and the rest are red. A disc is randomly selected from the bag. If it is white the process stops. If the disc is not white it is not returned to the bag and a second disc is randomly selected from the nine still in the bag. If this second disc is white the process stops. If this second disc is not white it is not returned to the bag and a third disc is randomly selected from the eight still in the bag. The process stops whatever colour this third disc is. Determine the probability that in this process
   (a) the 1st disc is not white,
   (b) 3 red discs are selected,
   (c) exactly 2 discs are selected,
   (d) 2 red and 1 white disc are selected,
   (e) 2 red and 1 white disc are selected, given more than one disc resulted.

Imagining the tree diagram.

Large tree diagrams can take time to construct so at times we may instead simply imagine following the various branches to an outcome, without actually drawing the tree diagram.



Example 10

A bag contains 10 marbles, 6 red and 4 blue. Four marbles are selected at random, one after the other, with each one selected <u>not</u> being returned to the bag before the next is selected.

Find the probability that this will produce

(a) 4 red marbles, (b) 3 reds then a blue, (c) 3 reds and 1 blue in any order.

"Thinking our way along the appropriate branches":

(a) $P(RRRR) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7}$

$= \frac{1}{14} \quad (\approx 0{\cdot}0714)$

(b) $P(3 \text{ red then blue}) = P(RRRB)$

$= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7}$

$= \frac{2}{21} \quad (\approx 0{\cdot}0952)$

(c) $P(3 \text{ red and } 1 \text{ blue}) = P(RRRB) + P(RRBR) + P(RBRR) + P(BRRR)$

$= \frac{2}{21} + \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} + \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7}$

$= \frac{8}{21} \quad (\approx 0{\cdot}3810)$

Example 11

Bag A contains 2 red discs and 3 blue discs.

Bag B contains 1 red disc and 4 blue discs.

A normal die is rolled once. If the outcome is a six a disc is selected from bag A, otherwise a disc is selected from bag B.

Find the probability that the selected disc is red.

"Thinking our way along the appropriate branches":

$P(\text{red disc}) = P(6 \text{ on the die then a red disc}) + P(\text{not } 6 \text{ then red disc})$

$= \frac{1}{6} \times \frac{2}{5} + \frac{5}{6} \times \frac{1}{5}$

$= \frac{7}{30}$

Probability rules.

Whilst drawing tree diagrams, Venn diagrams, making lists or constructing tables can help our understanding of probability questions and allow us to apply various probability rules intuitively we can apply such rules without the assistance of diagrams and lists if we wish to.

The probability rules are stated below.

Rule for P(A').

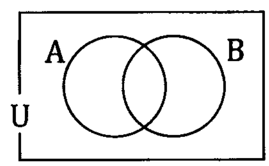
The Preliminary work section reminded us of the rule:

P(an event not occurring) = 1 - P(the event occurring)

Using A' or $\bar{A}$ to represent event A not occurring we can write this as

$$P(A') = 1 - P(A) \qquad \text{or} \qquad P(\bar{A}) = 1 - P(A)$$

Rule for P(B|A).



As we saw earlier in this chapter, consideration of the Venn diagram on the right leads to the statement:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Rule for P(A and B).

Rearranging the above rule leads to the **multiplication rule**:

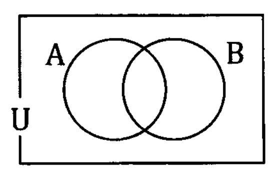
$$P(A \cap B) = P(A) \times P(B \mid A)$$

- Remember that $P(A \cap B)$ means the probability of A **and** B occurring.
- If the probability of B occurring is unaffected by whether or not A has occurred we say that events A and B are **independent**.

  In such cases $P(B|A) = P(B)$ and the multiplication rule becomes:

$$P(A \cap B) = P(A) \times P(B)$$

Rule for P(A or B).



As we saw earlier in this chapter, consideration of the Venn diagram on the right leads to the **addition rule**:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Remember that $P(A \cup B)$ means the probability of A **or** B occurring with the "or" meaning *at least one of.*
- If A and B are **mutually exclusive**, they cannot occur simultaneously. In such cases we have $P(A \cap B) = 0$ and the addition rule becomes:

$$P(A \cup B) = P(A) + P(B)$$

Summary

- Complementary events. (A and A'):

$$\mathbf{P(A') = 1 - P(A)}$$

- Conditional probability. (B|A):

$$\mathbf{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

- A **and** B. $(A \cap B)$:
  To determine the probability of A **and** B occurring we multiply the probabilities together, paying due regard to whether the occurrence of one of the events affects the likelihood of the other occurring:

$$\mathbf{P(A \cap B) = P(A) \times P(B \mid A)}$$

  If A and B are **independent** events, P(B|A) = P(B) and so

$$P(A \cap B) = P(A) \times P(B)$$

- A **or** B. $(A \cup B)$:
  To determine the probability of A **or** B occurring we add the probabilities together and then make the necessary subtraction to compensate for the "double counting of the overlap":

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

  If A and B are **mutually exclusive** events, $P(A \cap B) = 0$ and so

$$P(A \cup B) = P(A) + P(B)$$

These rules can be used to determine probabilities without first drawing tree diagrams, Venn diagrams etc., as the examples which follow will demonstrate. However, a note of caution is appropriate first:

Use the next few pages to gain familiarity with the probability rules but do not be too quick to forsake the various diagrammatic approaches in favour of a purely rules approach. Listing sample spaces, drawing tree diagrams and compiling Venn diagrams may take longer to do but such approaches can greatly clarify a problem, reduce errors and allow you to use the probability rules more intuitively, rather than "blindly" following a formula.

Example 12

The probability of a person having a particular disease is 0·001. If they have the disease the probability they will die from it is 0·7. What is the probability the person has the disease and will die from it?

Suppose event A is "has the disease" and event B is "will die from the disease".

We are given that P(A) = 0·001 and P(B | A) = 0·7 and we require P(A ∩ B).

Using
$$\begin{aligned} P(A \cap B) &= P(A) \times P(B \mid A) \\ P(A \cap B) &= 0{\cdot}001 \times 0{\cdot}7 \\ &= 0{\cdot}0007 \end{aligned}$$

The probability the person has the disease and will die from it is 0·0007.

Check that constructing a tree diagram also gives this answer.

Example 13

A box contains 50 items, five of which are defective. Two items are randomly chosen from the box, one after the other, the first not being replaced before the second is selected. What is the probability that the two selected will both be defective?

Applying the rule $P(A \cap B) = P(A) \times P(B \mid A)$:

$$\begin{aligned} P(\text{1st defective \& 2nd defective}) &= P(\text{1st defective}) \times P(\text{2nd defective} \mid \text{1st defective}) \\ &= \frac{5}{50} \times \frac{4}{49} \\ &= \frac{2}{245} \quad (\approx 0{\cdot}0082) \end{aligned}$$

Check that you can also obtain this answer using a tree diagram,

Example 14

The probability that in a particular piece of machinery component A will fail is 0·05. Provided A does not fail the probability that component B will fail is 0·01. However if A does fail then the probability of B failing rises to 0·1.

What is the probability that (a) A fails and B does not,

(b) A does not fail and B does not fail.

Given: P(A fails) = 0·05, P(B fails|A not fail) = 0·01, P(B fails|A fails) = 0·1.

(a)
$$\begin{aligned} P(\text{A fails and B not fail}) &= P(\text{A fails} \cap \text{B not fail}) \\ &= P(\text{A fails}) \times P(\text{B not fail} \mid \text{A fails}) \\ &= 0{\cdot}05 \times 0{\cdot}9 \\ &= 0{\cdot}045 \end{aligned}$$

(b)
$$\begin{aligned} P(\text{A not fail and B not fail}) &= P(\text{A not fail} \cap \text{B not fail}) \\ &= P(\text{A not fail}) \times P(\text{B not fail} \mid \text{A not fail}) \\ &= 0{\cdot}95 \times 0{\cdot}99 \\ &= 0{\cdot}9405 \end{aligned}$$

Check that constructing a tree diagram also gives these answers.

Example 15

Events A and B are such that P(A) = 0·65, P(B) = 0·6 and P(A ∩ B) = 0·4.
Determine (a) P(A ∪ B) (b) $P(\overline{A \cup B})$ (c) P(A | B).

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0{\cdot}65 + 0{\cdot}6 - 0{\cdot}4$
$= 0{\cdot}85$

(b) $P(\overline{A \cup B}) = 1 - P(A \cup B)$
$= 1 - 0{\cdot}85$
$= 0{\cdot}15$

(c) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
$= \dfrac{0{\cdot}4}{0{\cdot}6}$
$= \dfrac{2}{3}$

Check that using a Venn diagram also gives these answers.

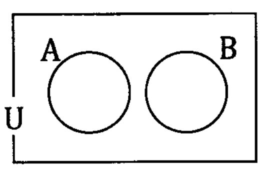
Example 16

A card is randomly drawn from a normal pack.
What is the probability that the card is (a) a red or a two, (b) a two or a jack.

(a) P(a red or a two) = P(red ∪ two)
= P(red) + P(two) − P(red ∩ two)
$= \dfrac{26}{52} + \dfrac{4}{52} - \dfrac{2}{52}$
$= \dfrac{7}{13}$

(b) P(a two or a Jack) = P(two ∪ jack)
= P(two) + P(jack) − P(two ∩ jack)
$= \dfrac{4}{52} + \dfrac{4}{52} - \dfrac{0}{52}$
$= \dfrac{2}{13}$

Notice that in the last example P(two ∩ jack) = 0 because the card we are selecting cannot be both a two and a jack. The events "card is a two" and "card is a jack" are **mutually exclusive.**



Independent events.

We know that for independent events we obtain the probability of A and B occurring by multiplying the separate probabilities together but how do we know if two events are independent?
In some situations we will intuitively know the events involved are independent. For example, if we roll a normal die and toss a coin, the outcome of the coin tossing is independent of the roll of the die. The probability of getting a head on the coin is 0·5 whatever the result of rolling the die. Similarly we would expect the result of tossing a

coin to be independent of the result of any previous toss. On the other hand, if we were selecting two marbles from a bag containing red and blue marbles, without replacing the first before the second is drawn, we know that probabilities associated with the colour of the second marble depend on the colour of the first marble.

Example 17
If a coin is flipped four times what is the probability of getting four heads?

The result of each flip of the coin is independent of previous flips.

Thus $\quad P(H\,H\,H\,H) = P(H) \times P(H) \times P(H) \times P(H)$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{16}$$

If the dependence or independence of two events is not obvious it may be stated in the question, as in the next example.

Example 18
Three students, Alex, Bill and Con, each take their driving test. Their instructor estimates the probability of each of them passing is as follows:

P(Alex passes) = 0·9, P(Bill passes) = 0·8, P(Con passes) = 0·6.

If these events are independent of each other determine the probability that

(a) Alex and Bill will pass but Con will not,

(b) All three people will pass.

(a) P(Alex & Bill pass & Con fails) = P(Alex pass) × P(Bill pass) × P(Con fail)
= 0·9 × 0·8 × 0·4
= 0·288

(b) P(All three people pass) = P(Alex pass) × P(Bill pass) × P(Con pass)
= 0·9 × 0·8 × 0·6
= 0·432

Example 19
A manufactured item consists of five parts, A, B, C, D and E. The probability of these parts being defective is 0·01, 0·2, 0·1, 0·02 and 0·01 respectively. The parts are manufactured by different companies so assume the occurrence of defective items are independent of each other. If we randomly select one of each item find the probability that all five are not defective.

P(all okay) = P(A okay) × P(B okay) × P(C okay) × P(D okay) × P(E okay)
= 0·99 × 0·8 × 0·9 × 0·98 × 0·99
≈ 0·7

Example 20
A die is rolled and a coin is tossed. Find the probability of obtaining a six on the die or a head on the coin.

$$\begin{aligned} P(6 \text{ or } H) &= P(6 \cup H) \\ &= P(6) + P(H) - P(6 \cap H) \\ &= P(6) + P(H) - P(6) \times P(H) \qquad \text{because P(6) and P(H) are independent} \\ &= \frac{1}{6} + \frac{1}{2} - \frac{1}{6} \times \frac{1}{2} \\ &= \frac{7}{12} \end{aligned}$$

The reader should confirm that the same answer can be obtained by creating a table of equally likely outcomes.

Exercise 9F. Initially attempt the following questions without drawing diagrams. Instead practise "imagining the tree diagram" and also "a rules approach".

Hint: Remember that if you know $P(A)$ then $P(A') = 1 - P(A)$.
Use of this rule can sometimes save a lot of time.

1. A bag contains 10 marbles: 6 red and 4 blue. Two marbles are randomly selected from the bag, the first not being replaced before the second is drawn. Determine the probability of getting
   (a) two reds, (b) two of the same colour,
   (c) no blues, (d) at least one blue.

2. A box contains 100 carburettors, four of which are faulty. A quality control person takes a carburettor from the box, tests it and then puts it to one side. This is repeated until four of the carburettors have been tested.
   Correct to four decimal places, what is the probability that
   (a) none of the four tested are faulty, (b) at least one of the four is faulty?

3. A box contains 200 carburettors, three of which are faulty. A quality control person takes a carburettor from the box, tests it and then puts it to one side. This is repeated until five of the carburettors have been tested.
   Correct to four decimal places, what is the probability that
   (a) none of the five tested are faulty, (b) at least one of the five is faulty?

4. A bag contains 10 marbles: 4 red, 3 blue and 3 green. A marble is randomly selected and not replaced, a second is randomly selected and not replaced, and then a third is randomly selected. Determine the probability of getting
   (a) three reds, (b) three of the same colour,
   (c) no reds, (d) at least one red.

5. A bag contains 10 marbles: 4 red, 3 blue and 3 green. Three marbles are randomly selected from the bag, one after the other, with each marble selected being put back into the bag before the next is selected. Determine the probability of getting
   (a) three reds, (b) three of the same colour,
   (c) no reds, (d) at least one red.

6. In a particular school the probability of a randomly selected student being in year 8 is 0·24.
   If a year 8 student is chosen at random the probability they are male is 0·52.
   What is the probability that a randomly selected student from this school is
   (a) a year 8 male,
   (b) a year 8 female?

7. Bag A contains 1 red disc and 3 blue discs.
   Bag B contains 2 red discs and 2 blue discs.
   A normal die is rolled once. If the outcome is a five or a six a disc is selected from bag A, otherwise a disc is selected from bag B.
   Find the probability that the selected disc is red.

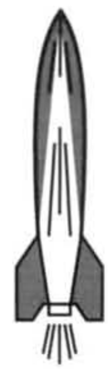
8. The probability of a randomly selected person having disease X is 0·001.
   A test is developed to detect whether or not a person has disease X.
   For people with the disease the test returns a positive result 98% of the time and wrongly returns a negative result for the other 2%.
   For people who do not have the disease the test returns a negative result 99% of the time and wrongly returns a positive result for the other 1%.
   If a person is chosen at random and given this test what is the probability they will return a positive result?

9. A fair coin is tossed and a normal die is rolled.
   Event A is that of the coin landing head uppermost.
   Event B is that of the uppermost face of the die showing a number less than 5.
   Determine (a) $P(A)$ (b) $P(B)$ (c) $P(A \cap B)$ (d) $P(A \cup B)$.

10. Two normal dice, one red and the other blue, are rolled.
    Event A is that of the uppermost face of the red die showing a number less than 3.
    Event B is that of the uppermost face of the blue die showing an even number.
    Determine (a) $P(A)$ (b) $P(B)$ (c) $P(A \cap B)$ (d) $P(A \cup B)$.

11. Events A and B are such that $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}5$ and $P(A \cap B) = 0{\cdot}1$.
    Determine (a) $P(A \cup B)$ (b) $P(\overline{A \cup B})$ (c) $P(A \mid B)$ (d) $P(B \mid A)$.

12. Events A and B are such that $P(A) = 0{\cdot}5$, $P(B) = 0{\cdot}8$ and $P(A \cup B) = 0{\cdot}9$.
    Determine (a) $P(A \cap B)$ (b) $P(A \mid B)$ (c) $P(B \mid A)$.



13. The fuel tank of a particular rocket has five seals, A, B, C, D and E that prevent leaks from the fuel tank. The likelihood of any one of these seals failing is independent of the behaviour of the other seals. During a launch these seals are under most strain and the engineers estimate that the probability of each seal failing is:
    P(A) = 0·02, P(B) = 0·2, P(C) = 0·15, P(D) = 0·01, P(E) = 0·005.
    Determine the probability that during a launch
    (a) all of the seals will fail,
    (b) none of the seals will fail, (answer rounded to two decimal places)
    (c) at least one will fail (answer rounded to two decimal places).

14. One child rolls a normal fair die and two other children, independently of each other, write down which number from 1 to 6 they think the die will show on its uppermost face. What is the probability that
    (a) both of these children guess correctly,
    (b) neither guess correctly,
    (c) at least one of them guess correctly?

15. Two components, X and Y, are manufactured independently of each other.
    For each type the probability of a randomly chosen component being defective is:
    P(X defective) = 0·005, P(Y defective) = 0·01.
    Determine the probability that for a randomly selected X and a randomly selected Y
    (a) both components are defective,
    (b) neither of the components are defective,
    (c) at least one of the components is defective.

16. Three components, X, Y and Z, are manufactured independently of each other. For each type the probability of a randomly chosen component being defective is:
    P(X defective) = 0·005, P(Y defective) = 0·01, P(Z defective) = 0·002.
    Determine the probability that if we randomly select one of each of these three components
    (a) all three components are defective,
    (b) none are defective, (round to 3 dp),
    (c) at least one is defective (round to 3 dp).

17. Bags A and B each contain five coloured discs. Bag A contains 3 green and 2 yellow. Bag B contains 1 green and 4 yellow.
    A normal die is rolled once and, if the result is even, one disc is randomly selected from bag A. If the result is odd one disc is randomly selected from bag B.
    Determine the probability of this process producing:
    (a) a disc from bag B,
    (b) a yellow disc from bag B,
    (c) a yellow disc,
    (d) a yellow disc or a disc from bag B.

18. Bags A and B each contain five coloured discs. Bag A contains 3 green and 2 yellow. Bag B contains 1 green and 4 yellow.
A normal die is rolled once and, if the result is greater than 4, one disc is randomly selected from bag A. If the result is not greater than 4 one disc is randomly selected from bag B.
Determine the probability of this process producing:
(a) a yellow disc from bag B (b) a yellow disc or a disc from bag B.

19. Events A and B are such that $P(A') = 0{\cdot}35$, $P(B) = 0{\cdot}34$ and $P(A \cup B) = 0{\cdot}86$.
Determine $P(A \cap B)$ and $P(B \mid A)$.

20. Events A and B are such that $P(B \mid A) = 0{\cdot}20$, $P(A \mid B) = 0{\cdot}25$,
and $P(A \cap B) = 0{\cdot}10$.
Determine $P(A \cup B)$.

21. Events A and B are such that $P(B|A) = \frac{1}{4}$, $P(A \mid B) = \frac{2}{5}$ and $P(A \cap B) = \frac{3}{22}$.
Determine $P(A \cup B)$.

More about independent events.

As mentioned earlier, for independent events A and B

$$P(B \mid A) = P(B)$$

and the rule $P(A \cap B) = P(A) \times P(B \mid A)$

reduces to: $P(A \cap B) = P(A) \times P(B)$.

It follows that if B is independent of A, then A is independent of B.

i.e. if $P(B \mid A) = P(B)$ then $P(A \mid B) = P(A)$, as proved below:

If $P(B \mid A) = P(B)$ then $P(A \cap B) = P(A) \times P(B)$.

But $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

and so $P(A \mid B) = \dfrac{P(A) \times P(B)}{P(B)}$

$= P(A)$ as required.

If A and B are independent events then	•	$P(B \mid A) = P(B)$
	•	$P(A \mid B) = P(A)$
	•	$P(A \cap B) = P(A) \times P(B)$

We can use these ideas as a test for independence:

If we can show that	•	$P(B \mid A) = P(B)$
or that	•	$P(A \mid B) = P(A)$
or that	•	$P(A \cap B) = P(A) \times P(B)$
then A and B are independent events.		

Example 21
Event A is that of rolling a die and getting an even number.
Event B is that of rolling a die and getting a number less than 5.
Prove that A and B are independent events.

$$P(A) = P(\text{even number}) = \frac{3}{6} = 0{\cdot}5$$

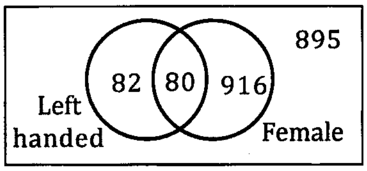
$$P(A|B) = P(\text{even number}|\text{a number} < 5) = \frac{2}{4} = 0{\cdot}5$$

Thus $P(A) = P(A \mid B)$ and hence A and B are independent.

The reader should confirm that for these events it is also the case that

$$P(B) = P(B \mid A) \quad \text{and} \quad P(A \cap B) = P(A) \times P(B).$$

Independence suggested.
If, with collected data, we were to find that $P(A) \approx P(A \mid B)$ this could suggest that events A and B are independent.
For example, suppose that a survey of one thousand nine hundred and seventy three individuals investigated, amongst other things, left handedness and gender, and gave rise to the Venn diagram shown on the right.



Based on these figures $\quad P(\text{Left handed}) = \frac{82 + 80}{1973} = 0{\cdot}082$

$$P(\text{Left handed} \mid \text{Female}) = \frac{80}{80 + 916} = 0{\cdot}080 \qquad P(\text{Left handed} \mid \text{Male}) = \frac{82}{82 + 895} = 0{\cdot}084$$

The closeness of these figures to each other indicates that whether a person is left handed could well be independent of gender. Approximately 8% of the entire group was left handed and this same percentage was seen within the males and the females.

More about mutually exclusive events.
We have already seen that if events A and B cannot both occur we say they are mutually exclusive. i.e. The occurrence of one of the events excludes the occurrence of the other.
Thus, for mutually exclusive events:

$$P(A \cap B) = 0$$

and the rule $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

reduces to: $\quad P(A \cup B) = P(A) + P(B).$

We can use this as a test for mutual exclusivity:

> If we can show that • $P(A \cap B) = 0$
> or that • $P(A \cup B) = P(A) + P(B)$
> then A and B are mutually exclusive events.

Example 22

If events A and B are such that P(A) = 0·4 and P(B) = $P(\overline{A \cup B})$= 0·3, prove that A and B are mutually exclusive.

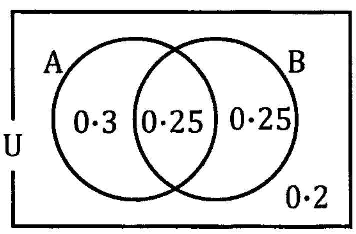
If $P(\overline{A \cup B}) = 0{\cdot}3$ then $P(A \cup B) = 0{\cdot}7$.
But $P(A) + P(B) = 0{\cdot}7$.
Thus $P(A \cup B) = P(A) + P(B)$ and hence A and B are mutually exclusive.

Alternatively the given probabilities can be used to complete a Venn diagram and it can be determined that $P(A \cap B) = 0$.

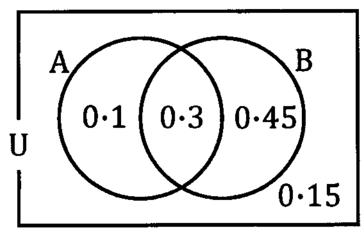
Exercise 9G

Each of the Venn diagrams below show the probabilities of events A and B occurring. In each case classify events A and B as either independent or dependent.

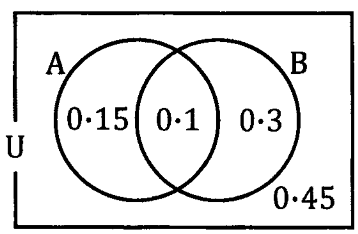
1.



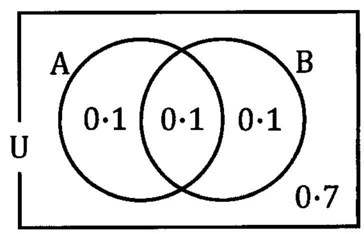
2.



3.

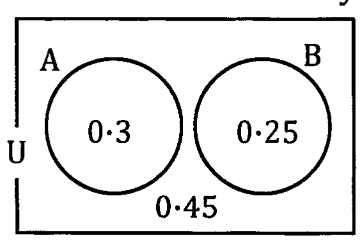


4.

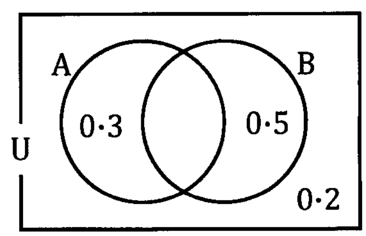


The numbers in the various sections of the following Venn diagrams indicate the probability of the event represented by that section occurring. In each case classify events A and B as either mutually exclusive or not mutually exclusive.

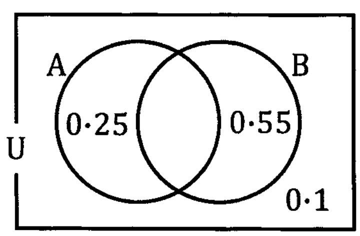
5.



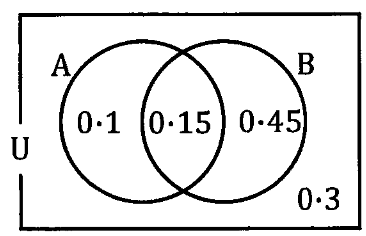
6.



7.



8.



9. The following pairs of events refer to one roll of a normal die. Which pairs of events are mutually exclusive?
   (a) obtaining a 3 and obtaining a 4,
   (b) obtaining an even number and obtaining a number less than 5,
   (c) obtaining a prime number and obtaining an even number
   (d) obtaining a number less than 3 and obtaining a number greater than 5.
   (e) obtaining a number less than 3 and obtaining a number less than 5.

10. A bag contains a number of marbles, some red and the rest blue.
    Two marbles are randomly selected from the bag, one after the other.
    Event A is that of the first marble being red.
    Event B is that of the second marble being red.
    State whether events A and B are dependent or independent if
    (a) the first marble is replaced before the second is selected,
    (b) the first marble is not replaced before the second is selected.

11. Earlier in this chapter the comment was made that for some situations we intuitively know that two events are independent, for example if we roll a normal die and toss a coin, we know that the outcome of the coin toss is independent of the roll of the die. However, even though the independence is intuitive, use the table of twelve equally likely outcomes shown below to confirm that
    $P(T) = P(T \mid 6)$ $\quad$ $P(6) = P(6 \mid T)$ $\quad$ $P(T \cap 6) = P(T) \times P(6)$

		DIE					
		1	2	3	4	5	6
COIN	Head	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
	Tail	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

12. Events A and B are independent events with $P(A) = 0{\cdot}2$ and $P(B) = 0{\cdot}25$.
    Determine (a) $P(A \mid B)$ (b) $P(B \mid A)$ (c) $P(A \cap B)$ (d) $P(\overline{A \cup B})$

13. Events A and B are mutually exclusive events with $P(A) = 0{\cdot}2$, $P(B) = 0{\cdot}3$.
    Determine (a) $P(A \cap B)$ (b) $P(B \mid A)$ (c) $P(A \mid B)$ (d) $P(A \cup B)$

14. Events A and B are such that $P(A) = 0{\cdot}25$, $P(B) = 0{\cdot}5$ and $P(\overline{A \cup B}) = 0{\cdot}25$.
    Prove that A and B are mutually exclusive.

15. Events A and B are independent events with $P(A) = 0{\cdot}5$ and $P(B) = 0{\cdot}6$.
    Determine (a) $P(A \cap B)$ (b) $P(A \cup B)$ (c) $P(B \mid A)$ (d) $P(A \mid B)$

16. Independent events A and B are such that $P(A \cup B) = 0{\cdot}85$ and $P(A) = 0{\cdot}25$.
    Find P(B).

17. Independent events A and B are such that $P(A \cup B) = 0{\cdot}4$ and $P(A) = 0{\cdot}25$.
    Find P(B).

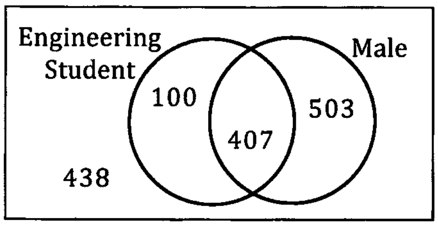
18. If $P(A) = 0{\cdot}2$ and $P(B) = 0{\cdot}4$ find $P(\overline{A \cup B})$ in each of the following cases:
    (a) A and B are mutually exclusive events,
    (b) A and B are independent events.

19. If P(A) = 0·2 and P(B) = 0·5 find $P(\overline{A \cup B})$ in each of the following cases:
    (a) A and B are mutually exclusive events,
    (b) A and B are independent events.

20. Let us suppose that for a particular activity the number of equally likely outcomes featuring or not featuring events A and B are as in the table on the right.
    Find x if (a) A and B are mutually exclusive,
    (b) A and B are independent.

	A	A'
B	x	5
B'	6	2

21. An analysis of the 1448 students at a college produced the figures shown in the Venn diagram below with regards to the numbers of males and females who were, or were not, taking one of the Engineering courses offered by the college (i.e. were or were not classified as an Engineering student).



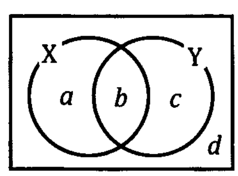
For a randomly chosen student from this college determine
(a) P(the student is an Engineering student),
(b) P(the student is an Engineering student | the student is male),
(c) P(the student is an Engineering student | the student is female).
(d) Comment on your results.

22. Final year students at a particular college can either follow the *Normal* course in their chosen subject or, if their grades in the previous years have been high enough, they can follow the *Honours* course in that subject. The table below shows the distribution of male and female final year students across these two levels.

	Normal course	Honours course	Totals
Female	2814	1540	4354
Male	1916	982	2898
Totals	4730	2522	7252

For a randomly chosen final year student from this college determine
(a) P(the student is on the honours course),
(b) P(the student is on the honours course | the student is male),
(c) P(the student is on the honours course | the student is female).
(d) Comment on your results.

23. The probabilities of events X and Y occurring or not are as in the Venn diagram on the right, $a + b + c + d = 1$.
Use a, b, c and d to prove that if P(X) = P(X | Y) then it follows that P(Y) = P(Y | X) and P(X ∩ Y) = P(X) P(Y).



Miscellaneous Exercise Nine.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Events A and B are such that $P(A) = 0{\cdot}7$, $P(B) = 0{\cdot}6$ and $P(A \cup B) = 0{\cdot}8$.

   Determine (a) $P(A \cap B)$ (b) $P(\overline{A \cap B})$ (c) $P(A \mid B)$
   (d) $P(\overline{A} \mid B)$ (e) $P(A \mid \overline{B})$

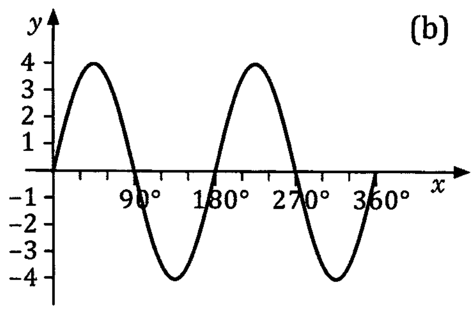
2. Events A and B are such that $P(A) = 0{\cdot}45$, $P(B) = 0{\cdot}2$ and $P(A \cup B) = 0{\cdot}56$.
   Prove that A and B are independent events.

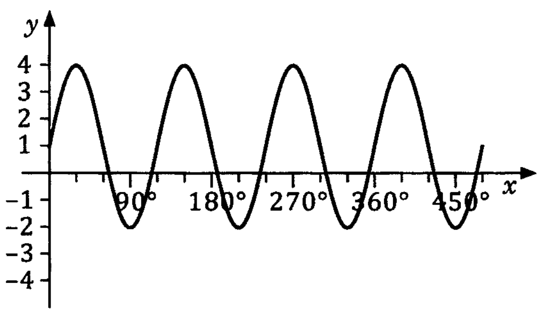
3. Solve the quadratic equation $2x^2 - x - 36 = 0$ three times:

   Once using the method of completing the square.
   Once using the quadratic formula.
   Once using factorisation.

4. A child has four lollies in a bag. Two of the lollies are red, one is green and one is yellow. The child eats the lollies one by one, each time selecting the next one to eat randomly.
   Determine the probability that the third lolly the child eats is red in each of the following situations.
   (a) The first one the child eats is green and the second one is yellow.
   (b) The first one the child eats is red.
   (c) Nothing is known about the order the lollies are eaten.

5. State the period and amplitude of each of the following.

   (a)


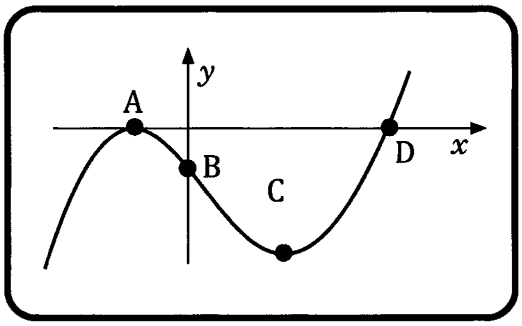
   (b)


6. The smallest positive value of x for which $\sin x° = 0{\cdot}53$ is $x = 32$, to the nearest integer. Without the assistance of your calculator find to the nearest integer all values of x in the interval $-360 \le x \le 360$ for which $\sin x° = -0{\cdot}53$.

7. Point B (5, −2) is the mid point of the line joining point A (3, −5) to point C.
   Find the coordinates of point C.

8. A company employs 93 people of whom 38 are male.
   Twenty two of the employees walk to work and 15 of these 22 are female.
   If one of the 93 employees is chosen at random determine the probability that they are
   (a) female,
   (b) a male who walks to work,
   (c) male given they walk to work,
   (d) someone who walks to work given they are male.

9. The diagram on the right shows a sketch of $y = (x + 2)^2 (x - 7)$.



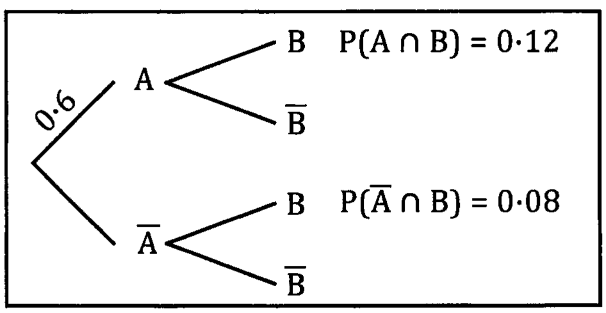
   (a) Without using a calculator find the coordinates of point B, the y-axis intercept,
   (b) Without using a calculator find the coordinates of points A and D, the x-axis intercepts.
   (c) The minimum turning point, C, has coordinates (a, b) where a and b are both integers. Use a graphic calculator to determine a and b.
   (d) Determine the range of values of p for which the equation $(x + 2)^2 (x - 7) = p$ has three distinct solutions.

10. If we assume that when $\theta = \frac{\pi}{13}$ radians then $\sin\theta = 0{\cdot}24$,
determine solutions to the equation
$$(6 + 25\sin\theta)(1 - 2\cos\theta) = 0 \qquad \text{for } 0 \le \theta \le 2\pi.$$

11. Solve this question three times - once using a tree diagram approach,
once using a Venn diagram approach,
once using a rules approach.

In a class of thirty students the teacher is surprised to find that two of the sixteen boys and five of the girls are left handed. (None of the thirty students are ambidextrous.)
If one of these thirty students is chosen at random determine the probability that the chosen student is
(a) a left handed boy, (b) a right handed girl,
(c) left handed given that the chosen student is a girl,
(d) a girl given that the chosen student is left handed.

12. The first stage of a two part random process involves the occurrence, or non occurrence of outcome A. For each of these eventualities the second stage then involves the occurrence, or non occurrence of outcome B. The probabilities associated with some of these events are shown in the tree diagram on the right.



Determine (a) $P(\overline{A} \cap \overline{B})$ (b) $P(B)$ (c) $P(A \cup B)$
(d) $P(B|A)$ (e) $P(A|B)$
(f) Are events A and B independent? (Justify your answer.)